Self-sufficient sets in smartgrids

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Abstract

We consider the problem of finding microgrids in a network. We mainly focus on the complexity aspects related to the different variants of this problem[5].

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1 Introduction and motivations

The electric landscape in France is in deep mutation. The electric production is changing, moving from a small number of production plants with high electric power to a huge number of production units, each delivering a small electric power. From a legal point of view, it is now possible since 2017 to gather consumers and producers in a private local network called *microgrid*. In such microgrids, the consumers use the electricity generated by the producers belonging to this microgrid. The only electric exchange between a microgrid and the outside is the one necessary to obtain the equilibrium between electric consumption and production of the whole microgrid.

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The merging of consumers and producers geographically close into micro-grids presents several advantages. The transportation of electricity is more efficient in a microgrid as producers and consumers are close. There may exist local rules regulating the production of electricity such as 100% of renewable energy. Moreover, the partition of the whole electric network into microgrids tends to reduce the effects of electric problems such as blackouts. Indeed, when a problem locally appears in a microgrid, this latter can be disconnected from the network stopping the propagation of the problem.

A microgrid is interesting only if the exchange between this microgrid and the outside is very small. Ideally, the local production should correspond to the local consumption. Moreover, the electric network of the microgrid must be sufficient to ensure the transportation of the electricity inside the microgrid.

We model the electrical network (smartgrid) with a graph $G = (V, E)$ whose vertices represent the producers/consumers/relays and whose edges are the electrical connections. We associate with each vertex $v \in V$ a weight $w \in \mathbb{Z}$ which corresponds to its consumption ($w < 0$) or production ($w > 0$). We think of microgrids as connected subgraphs, and call a microgrid *self-sufficient* if its own production matches its consumption. More formally, a subset of vertices X of V is called *self-sufficient* if the graph $G[X]$ induced by X is connected and its own consumption matches its production, that is, $w(X) = 0$.

We consider in this work two problems relative with the question of finding microgrids in a network. In the first one – hereafter called SELF-SUFFICIENT PARTITION – an electricity supplier has its own electrical network and wants to partition it into independent microgrids in order to limit the propagation of local electric problems. This problem can be formally defined as follows.

SELF-SUFFICIENT PARTITION. Given a graph $G = (V, E)$, $w : V \to \mathbb{Z}$ and $k \in \mathbb{Z}_+$, determine whether there exists a partition $\mathcal{P} = \{P_1, \dots, P_k\}$ of V such that P_i is self-sufficient for all $i = 1, \dots, k$.

In the second problem we consider – called SELF-SUFFICIENT AUGMENTATION – a set of local producers and consumers want to gather together to create a microgrid. A definition in terms of graphs in given below.

SELF-SUFFICIENT AUGMENTATION. Given a graph $G = (V, E)$, $w : V \to \mathbb{Z}$ and $X \subseteq V$, determine whether there exists a self-sufficient subset Y of V containing X.

In this work, we first state the complexity of SELF-SUFFICIENT PARTITION depending on the class graph of G. More precisely, we show that the problem is NP-complete whenever G is series-parallel but becomes polynomial-time solvable if G is outerplanar.

We then study the complexity of Self-Sufficient Augmentation. We show that it is NP-complete even if G is a star. We then consider its combinatorial version and prove that it is NP-complete in general but polynomial for graphs of fixed treewidth.

2 Complexity of Self-Sufficient Partition

In this section, we prove that the problem of partitioning a series-parallel graph into a fixed number of self-sufficient sets is NP-complete but becomes polynomial-time solvable in outerplanar graphs. Outerplanar graphs being a large subclass of series-parallel graphs, these two results, Theorems 2.1 and 2.2, establish the complexity behaviour of Self-Sufficient Partition.

2.1 NP-completeness

In this section, we prove the NP-completeness of Self-Sufficient Partition using the well-known NP-complete problem Partition [2] defined as follows.

Partition. Given a multiset of positive integers $p_1, \ldots, p_n$, determine whether there exists a partition of $\{1, \ldots, n\}$ into two subsets S_1 and S_2 such that $\sum_{i \in S_1} p_i = \sum_{i \in S_2} p_i$.

A graph is *series-parallel* if it does not contain K_4 as a minor. When no removal of a single vertex disconnects a graph, the latter is said 2-connected. Loops and bridges are called trivial 2-connected graphs. The non trivial 2-connected components of a graph are the maximal 2-connected subgraphs of the graph obtained after the removal of loops and bridges. Series-parallel graphs admit the following constructive characterization: a graph is series-parallel if all its non trivial 2-connected components can be built, starting from the circuit of length two, by repeatedly applying the following operations: add a parallel edge to an existing edge; or subdivide an existing edge, that is replace the edge by a path of length two.

Theorem 2.1 Self-Sufficient Partition *is NP-complete even if* $k = 2$ *and* G *is a* 2*-connected series-parallel graph.*

Proof. We reduce Partition to Self-Sufficient Partition. Let $p_1, \ldots, p_n$ be an instance of Partition and define $q = \frac{1}{2} \sum_{i=1}^{n} p_i$. Let $G = (V, E)$ be the graph with $n + 2$ vertices $s, t, v_1, \ldots, v_n$ and the $2n$ edges sv_i and

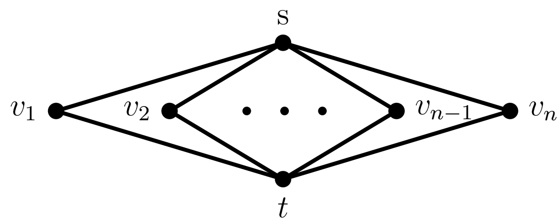

Fig. 1. Reduction from Partition to Self-Sufficient Partition.

$v_i t$ for $i = 1, \ldots, n$ (See Figure 1). Note that G is series-parallel. Let $w(s) = w(t) = -q$ and $w(v_i) = p_i$.

Let $\{P_1, P_2\}$ be a solution to Self-Sufficient Partition. By construction, if s and t belong to the same P_i, then $G[V \setminus P_i]$ is not connected. Hence, wlog, $s \in P_1$ and $t \in P_2$. Therefore, there exists $I \subseteq \{1, \ldots, n\}$ such that $P_1 = \{s\} \cup \{v_i : i \in I\}$ and $P_2 = \{t\} \cup \{v_i : i \notin I\}$. As P_1 and P_2 are self-sufficient, $w(P_1) = w(P_2) = 0$. Since $w(s) = w(t) = -q$, we have $w(\{v_i : i \in I\}) = \sum_{i \in I} p_i = q$ and $w(\{v_i : i \notin I\}) = \sum_{i \notin I} p_i = q$. Therefore, $\{I, \{1, \ldots, n\} \setminus I\}$ is a solution to Partition.

Conversely, if $\{S_1, S_2\}$ is a solution to Partition, then $V_1 = \{s\} \cup \{v_i : i \in S_1\}$ and $V_2 = \{t\} \cup \{v_j : j \in S_2\}$ are both self-sufficient and form a partition of V, hence $\{V_1, V_2\}$ is a solution to Self-Sufficient Partition. □

2.2 Polynomial case

We prove that if the desired number of self-sufficient sets of the partition is fixed, and if the graph is outerplanar, then Self-Sufficient Partition can be solved in polynomial time. A graph is *outerplanar* if it can be drawn on the plane so that all its vertices belong to the external face. Equivalently, a graph is outerplanar if it contains neither K_4 nor $K_{2,3}$ as a minor. Recall that series-parallel graphs are the graphs with no K_4-minor. The following result and Theorem 2.1 establish the complexity boundary of Self-Sufficient Partition.

Theorem 2.2 *If G is 2-connected outerplanar and k is fixed, then* Self-Sufficient Partition *is polynomial-time solvable.*

Proof. Let G be outerplanar and 2-connected and k be fixed. The goal is to find a partition of G into k self-sufficient sets of vertices. Let C be the cycle which forms the external face of G. By the following claim, enumeration gives an algorithm that runs in $O(n^{2k})$.

Claim 2.3 *If $\mathcal{P}$ is a self-sufficient partition of G, then there exists $P \in \mathcal{P}$ such that the vertices of P form a subpath of C.*

Proof. Suppose not, then there exist distinct P and Q in $\mathcal{P}$ such that C traverses, in this order, a set of vertices X_P of P, a set of vertices X_Q of Q, a set of vertices Y_P of P, and then a set of vertices Y_Q of Q. Since both $G[P]$ and $G[Q]$ are connected, we may assume that $G[X_P \cup Y_P]$ and $G[X_Q \cup Y_Q]$ contain an edge e_P and e_Q of $E \setminus C$, respectively. But then e_P and e_Q are crossing, a contradiction to the fact that G is outerplanar with external face C. $\square$

Now, note that there are $\binom{n}{2}$ subpaths of C. Let P be the vertex set of such a path and let $G' = G \setminus P$. The addition of P to any partition of G' into $k-1$ self-sufficient sets yields a partition of G into k self-sufficient sets. Since repeating this process decreases k by one, and since there are at most $\binom{n}{2} \leq n^2$ subpaths at each step, all the solutions are enumerated in less than $(n^2)^k$ operations. In particular, if k is fixed, then this is polynomial in n. $\square$

3 Complexity of Self-Sufficient Augmentation

3.1 Weighted version

We prove that Self-Sufficient Augmentation is NP-complete by reducing Subset Sum to it. Subset Sum is a well-known NP-complete problem, see [2].

Subset Sum. Given a multiset of integers $p_1, \dots, p_n$ and $q \in \mathbb{Z}$, determine whether there exists a subset I of $\{1, \dots, n\}$ such that $\sum_{i \in I} p_i = q$.

Theorem 3.1 Self-Sufficient Augmentation *is NP-complete even if G is a star.*

Proof. We reduce Subset Sum to Self-Sufficient Augmentation. Let $p_1, \dots, p_n$ and q be an instance of Subset Sum. Wlog, we assume that p_i is nonzero for $i = 1, \dots, n$. Let $G = (V, E)$ be the graph with $n+1$ vertices $s, v_1, \dots, v_n$ and the n edges sv_i for $i = 1, \dots, n$ (See Figure 2). Note that G is a star. Define w by $w(s) = -q$ and $w(v_i) = p_i$, and let $X = \{s\}$. First, note that if $\sum_{i \in I} p_i = q$ for some subset I of $\{1, \dots, n\}$, then $\{s\} \cup \{v_i : i \in I\}$ is self-sufficient. Conversely, let us show that any self-sufficient subset Y of V induces a solution to Subset Sum. By construction, since $G[Y]$ is connected, then $s \in Y$. Hence, $Y = \{s\} \cup \{v_i : i \in I\}$ for some $I \subseteq \{1, \dots, n\}$. By definition, $w(Y) = 0$ so $w(Y \setminus \{s\}) = \sum_{i \in I} p_i = -w(s) = q$. Thus, I is a solution to Subset Sum. $\square$

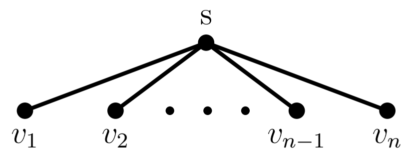

Fig. 2. Reduction from Subset Sum to Self-Sufficient Augmentation.

3.2 Combinatorial version

Given the difficulty of Self-Sufficient Augmentation shown in Theorem 3.1, we now consider the combinatorial variant of this problem. In this version, the vertices of the graph are either producers or consumers, and their production/consumption is not taken into account: the goal of a microgrid is to have as many producers as consumers.

A *bicolored graph* is a pair (G, π) where $G = (V, E)$ is an undirected graph and $\pi = \{V_1, V_2\}$ is a bipartition of V representing the color of each vertex. The vertices of V_1 are colored in red and those of V_2 in blue. A subgraph $G' = (V', E')$ of a bicolored graph (G, π) is *self-sufficient* if it is connected and V' contains the same number of vertices of each class of π. This is the special case of the weighted version of self-sufficiency when w takes only $+1$ and -1 values. Self-Sufficient Augmentation is defined in its combinatorial version as follows.

Combinatorial Self-Sufficient Augmentation. Given a bicolored graph (G, π), a subset W of vertices and $k \in \mathbb{Z}_+$, determine whether there exists a self-sufficient subgraph $G' = (V', E')$ of G such that V' contains W and $|V'| \leq k$.

3.2.1 Positive results

We prove that Combinatorial Self-Sufficient Augmentation reduces to Graph Motif. The latter problem can be stated as follows.

Graph Motif. Given a colored graph $G = (V, E)$ and a multiset of colors M, determine whether there exists a subset $X \subseteq V$ which induces a connected graph and whose multiset of colors equals M.

Proposition 3.2 Combinatorial Self-Sufficient Augmentation *reduces to* Graph Motif.

Proof. Consider an instance of Combinatorial Self-Sufficient Augmentation given by a graph $G = (V, E)$, a partition $\pi = \{V_1, V_2\}$ of V, a vertex subset of $W \subseteq V$ and a positive integer k. We suppose wlog that

$|W \cap V_1| \leq |W \cap V_2|$ and set $d = |W \cap V_2| - |W \cap V_1|$. Any self-sufficient subgraph will contain at least $|W| + d$ vertices. Let $\ell = \lfloor \frac{k-|W|-d}{2} \rfloor$. A solution to COMBINATORIAL SELF-SUFFICIENT AUGMENTATION – if it exists – is a self-sufficient subgraph containing $|W|+d+2j$ vertices for some $j \in \{0, \ldots, \ell\}$.

We define an instance $(\tilde{G}, \tilde{M})$ of GRAPH MOTIF as follows. Consider $\ell+1$ copies of G to define $\tilde{G}$. Its vertices are colored in blue and red according to the color of the vertices in G. Moreover, in each copy, all the vertices of W are colored in a new color, say green. Let v_0 be a vertex of W. For the j^{th} copy of G ($j \in \{0, \ldots, \ell\}$), add the path $P_j = v, u_1, w_1, \ldots, u_{\ell-j}, w_{\ell-j}, v_1$ where all the vertices but v are new vertices, v being v_0 in this copy of G. the vertices $u_1, \ldots, u_j$ are colored in red, $w_1, \ldots, w_j$ in blue and v_1 in green.

Let $\tilde{M}$ be the multiset of colors defined as follows. It contains $|W| + 1$ times the green color, $\ell + d$ times the red color and ℓ times the blue color.

Now, let $G' = (V', E')$ be a solution to COMBINATORIAL SELF-SUFFICIENT AUGMENTATION with $|V'| = |W|+d+2j$, $j \in \{0, \ldots, \ell\}$. A solution to GRAPH MOTIF is obtained by taking all the vertices corresponding to those of V' in the j^{th} copy of G plus the vertices of P_j.

Let W be a solution to GRAPH MOTIF. By construction, all the vertices of W belong to a same copy of G. Removing the vertices of the added path in W except the one associated with v_0 provides a solution to COMBINATORIAL SELF-SUFFICIENT AUGMENTATION. □

Proposition 3.2 implies that the polynomial cases for GRAPH MOTIF are also polynomial cases for COMBINATORIAL SELF-SUFFICIENT AUGMENTATION. Since GRAPH MOTIF is polynomial when there is a polynomial number of colors and G is of bounded treewidth [1], we obtain the following result.

Corollary 3.3 COMBINATORIAL SELF-SUFFICIENT AUGMENTATION *is polynomial-time solvable if G has a fixed treewidth.*

More precisely, if w denotes the treewidth of G, one can solve COMBINATORIAL SELF-SUFFICIENT AUGMENTATION in $O(|V|^{4w+2})$.

3.2.2 Negative results

We show a NP-hardness result for COMBINATORIAL SELF-SUFFICIENT AUGMENTATION. It is similar to the one existing for GRAPH MOTIF. The proof is based on the one of [1]. The proof uses a reduction from EXACT COVER BY 3-SETS which is NP-complete [2].

EXACT COVER BY 3-SETS. Given a set $X = \{x_1, x_2, \ldots, x_{3q}\}$ and a collection $S = \{s_1, s_2, \ldots, s_n\}$ of 3-element subsets of X, determine whether there exists a sub-collection $C \subseteq S$ such that every element of X is included in exactly one subset $s_i \in C$.

Proposition 3.4 COMBINATORIAL SELF-SUFFICIENT AUGMENTATION *is NP-hard even if G bipartite with maximum degree less than or equal to four.*

Proof. We slightly modify the reduction given in Theorem 2 of [1] from EXACT COVER BY 3-SETS to GRAPH MOTIF with two colors and G bipartite with maximum degree four. In their proof, the authors construct, from an instance X, S of EXACT COVER BY 3-SETS, a bipartite graph G containing $2n + q$ white vertices and n black ones. The motif M to find is composed of $2n + 3q$ white vertices and q black ones.

We construct from G a bipartite G' by replacing edge $s'_n s''_n$ by a path $s'_n, v_1, v_2, \ldots, v_{2n+2q}, s''_n$, where $v_1, \ldots, v_{2n+2q}$ are new black vertices. We set W equal to the set of white vertices and $k = |V|$. There exists a motif M in G if and only if there exists a self-sufficient subgraph of G'. □

Note that the proof may be adapted for an arbitrary value of k by adding dummy nodes.

4 Conclusion

We provide complexity results related to finding microgrids in an electric network. However, real-world applications involve more realistic considerations. For instance, the level of electric production/consumption cannot be considered as a single number; one needs to consider production/consumption profiles depending on time or stochastic variability instead. Furthermore, connectivity is not enough to ensure the transportation of electricity within the microgrid. Tacking into account these more realistic requirements is the direction of our future work.

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